Recurrence Relations

Evaluating Terms and Limits

2000 P1

A5. Two sequences are generated by the recurrence relations $u_{n+1} = au_n + 10$ and $v_{n+1} = a^2v_n + 16$.

The two sequences approach the same limit as $n \to \infty$.

Determine the value of a and evaluate the limit. **5**

3. (JAN) 02 P2

(*a*) Calculate the limit as n $\to \infty$ of the sequence defined by $u_{n+1} = 0{\cdot}9u_n + 10$, $u_0 = 1$. **3**

(*b*) Determine the least value of n for which u_n is greater than half of this limit and the corresponding value of u_n. **2**

2004 P2

4. A sequence is defined by the recurrence relation $u_{n+1} = ku_n + 3$.

(*a*) Write down the condition on k for this sequence to have a limit. **1**

(*b*) The sequence tends to a limit of 5 as $n \to \infty$. Determine the value of k. **3**

2005 P1

6. (*a*) The terms of a sequence satisfy $u_{n+1} = ku_n + 5$. Find the value of k which produces a sequence with a limit of 4. **2**

(*b*) A sequence satisfies the recurrence relation $u_{n+1} = mu_n + 5$, $u_0 = 3$.

(i) Express u_1 and u_2 in terms of m.

(ii) Given that $u_2 = 7$, find the value of m which produces a sequence with no limit. **5**

2006 P1

4. A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}8u_n + 12$, $u_0 = 4$.

(*a*) State why this sequence has a limit. **1**

(*b*) Find this limit. **2**

2007 P1

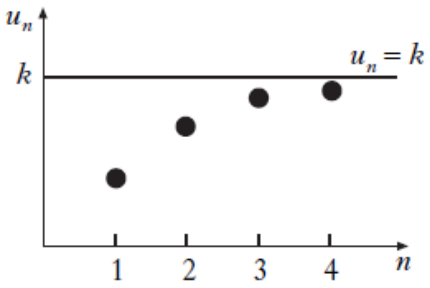
7. A sequence is defined by the recurrence relation

$$u_{n+1} = \frac{1}{4}u_n + 16,\ u_0 = 0.$$

(*a*) Calculate the values of u_1, u_2 and u_3. **3**

Four terms of this sequence, u_1, u_2, u_3 and u_4 are plotted as shown in the graph.

As $n \to \infty$, the points on the graph approach the line $u_n = k$, where k is the limit of this sequence.



(*b*) (i) Give a reason why this sequence has a limit.

(ii) Find the exact value of k. **3**

2015 OLD P2

3. A version of the following problem first appeared in print in the 16th Century.

A frog and a toad fall to the bottom of a well that is 50 feet deep.

Each day, the frog climbs 32 feet and then rests overnight. During the night, it slides down $\frac{2}{3}$ of its height above the floor of the well.

The toad climbs 13 feet each day before resting.

Overnight, it slides down $\frac{1}{4}$ of its height above the floor of the well.

Their progress can be modelled by the recurrence relations:

- $f_{n+1} = \frac{1}{3} f_n + 32, \quad f_1 = 32$
- $t_{n+1} = \frac{3}{4} t_n + 13, \quad t_1 = 13$

where f_n and t_n are the heights reached by the frog and the toad at the end of the nth day after falling in.

(*a*) Calculate t_2, the height of the toad at the end of the second day. **1**

(*b*) Determine whether or not either of them will eventually escape from the well. **5**

2015 P2

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where f_n and t_n are the heights reached by the frog and the toad at the end of the nth day after falling in.

(a) Calculate t_2, the height of the toad at the end of the second day. **1**

(b) Determine whether or not either of them will eventually escape from the well. **5**

2016 P1	**3.** A sequence is defined by the recurrence relation $u_{n+1} = \frac{1}{3}u_n + 10$ with $u_3 = 6$. (a) Find the value of u_4. 1 (b) Explain why this sequence approaches a limit as $n \to \infty$. 1 (c) Calculate this limit. 2

Pre 2000 - Evaluating Terms and Limits

1	A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}3u_n + 5$ with first term u_1. (*a*) Explain why this sequence has a limit as n tends to infinity. 1 (*b*) Find the **exact** value of this limit. 2
2	A sequence is defined by the recurrence relation $u_n = 0{\cdot}9u_{n-1} + 2$, $u_1 = 3$. (*a*) Calculate the value of u_2. 1 (*b*) What is the smallest value of n for which $u_n > 10$? 1 (*c*) Find the limit of this sequence as $n \to \infty$. 2
3	Two sequences are defined by these recurrence relations: $u_{n+1} = 3u_n - 0.4$ with $u_0 = 1$, $v_{n+1} = 0.3v_n + 4$ with $v_0 = 1$. (*a*) Explain why only one of these sequences approaches a limit as $n \to \infty$. 1 (*b*) Find algebraically the exact value of the limit. 2 (*c*) For the other sequence, find (i) the smallest value of n for which the n^{th} term exceeds 1000, and (ii) the value of that term. 2
4	Two sequences are defined by the recurrence relations $u_{n+1} = 0{\cdot}2u_n + p$, $u_0 = 1$ and $v_{n+1} = 0{\cdot}6v_n + q$, $v_0 = 1$. If both sequences have the same limit, express p in terms of q. 3

Finding a and b

2003 P1	**4.** A recurrence relation is defined by $u_{n+1} = pu_n + q$, where $-1 < p < 1$ and $u_0 = 12$. (*a*) If $u_1 = 15$ and $u_2 = 16$, find the values of p and q. **2** (*b*) Find the limit of this recurrence relation as $n \to \infty$. **2**
2011 P2	**3.** (*a*) A sequence is defined by $u_{n+1} = -\frac{1}{2}u_n$ with $u_0 = -16$. Write down the values of u_1 and u_2. **1** (*b*) A second sequence is given by 4, 5, 7, 11, ... It is generated by the recurrence relation $v_{n+1} = pv_n + q$ with $v_1 = 4$. Find the values of p and q. **3** (*c*) Either the sequence in (*a*) or the sequence in (*b*) has a limit. (i) Calculate this limit. (ii) Why does the other sequence not have a limit? **3**

2013 P2

1. The first three terms of a sequence are 4, 7 and 16.

The sequence is generated by the recurrence relation

$$u_{n+1} = mu_n + c, \text{ with } u_1 = 4.$$

Find the values of m and c. **4**

2015 SP P2

2. A wildlife reserve has introduced conservation measures to build up the population of an endangered mammal. Initially the reserve population of the mammal was 2000. By the end of the first year there were 2500 and by the end of the second year there were 2980.

It is believed that the population can be modelled by the recurrence relation:

$u_{n+1} = au_n + b$,

where a and b are constants and n is the number of years since the reserve was set up.

(a) Use the information above to find the values of a and b. **4**

(b) Conservation measures will end if the population stabilises at over 13 000. Will this happen? Justify your answer. **3**

2015 EP P2

1. A sequence is defined by $u_{n+1} = -\frac{1}{2}u_n$ with $u_0 = -16$.

(a) Determine the values of u_1 and u_2. **1**

(b) A second sequence is given by 4, 5, 7, 11, ...

It is generated by the recurrence relation $v_{n+1} = pv_n + q$ with $v_1 = 4$.

Find the values of p and q. **3**

(c) Either the sequence in (a) or the sequence in (b) has a limit.

(i) Calculate this limit.

(ii) Why does this other sequence not have a limit? **3**

2017 P1

9. A sequence is generated by the recurrence relation $u_{n+1} = mu_n + 6$ where m is a constant.

(a) Given $u_1 = 28$ and $u_2 = 13$, find the value of m. **2**

(b) (i) Explain why this sequence approaches a limit as $n \to \infty$. **1**

(ii) Calculate this limit. **2**

Extended Questions

2001 P2	**3.** On the first day of March, a bank loans a man £2500 at a fixed rate of interest of 1·5% per month. This interest is added on the last day of each month and is calculated on the amount due on the first day of the month. He agrees to make repayments on the first day of each subsequent month. Each repayment is £300 except for the smaller final amount which will pay off the loan. (*a*) The amount that he owes at the start of each month is taken to be the amount still owing just after the monthly repayment has been made. Let u_n and u_{n+1} represent the amounts that he owes at the starts of two successive months. Write down a recurrence relation involving u_{n+1} and u_n. **2** (*b*) Find the date and the amount of the final payment. **4**
2002 P2	**4.** A man decides to plant a number of fast-growing trees as a boundary between his property and the property of his next door neighbour. He has been warned, however, by the local garden centre that, during any year, the trees are expected to increase in height by 0·5 metres. In response to this warning he decides to trim 20% off the height of the trees at the start of any year. (*a*) If he adopts the "20% pruning policy", to what height will he expect the trees to grow in the long run? **3** (*b*) His neighbour is concerned that the trees are growing at an alarming rate and wants assurances that the trees will grow no taller than 2 metres. What is the minimum percentage that the trees will need to be trimmed each year so as to meet this condition? **3**
2008 SP P1	**21.** A firm cleans the factory floor on a daily basis with disinfectant. It has a choice of two products, either "A" or "B". Product A removes 70% of all germs but during the next 24 hours, 300 "new" germs per sq unit are estimated to appear. Product B removes 80% of all germs but during the next 24 hours, 350 "new" germs per sq unit are estimated to appear. For product A, let u_n represent the number of germs per sq unit on the floor immediately before disinfecting for the nth time. For product B, let v_n represent the number of germs per sq unit on the floor immediately before disinfecting for the nth time. (*a*) Write down a recurrence relation for each product to show the number of germs per sq unit present prior to disinfecting. **2** (*b*) Determine which product is more effective in the long term. **4**

Pre 2000 Questions - Extended Questions

1	A gardener feeds her trees weekly with "Bioforce, the wonder plant food". It is known that in a week the amount of plant food in the tree falls by about 25%. (*a*) The trees contain no Bioforce initially and the gardener applies 1g of Bioforce to each tree every Saturday. Bioforce is only effective when there is continuously more than 2g of it in the tree. Calculate how many weekly feeds will be necessary before the Bioforce becomes effective. **(3)** (*b*) (i) Write down a recurrence relation for the amount of plant food in the tree immediately after feeding. **(1)** (ii) If the level of Bioforce in the tree exceeds 5g, it will cause leaf burn. Is it safe to continue feeding the trees at this rate indefinitely? **(4)**
2	The sum of £1000 is placed in an investment account on January 1st and, thereafter, £100 is placed in the account on the first day of each month. • Interest at the rate of 0·5% per month is credited to the account on the last day of each month. • This interest is calculated on the amount in the account on the first day of the month. (*a*) How much is in the account on June 30th ? **(4)** (*b*) On what date does the account first exceed £2000? **(2)** (*c*) Find a recurrence relation which describes the amount in the account, explaining your notation carefully. **(3)**
3	Trees are sprayed weekly with the pesticide, KILLPEST, whose manufacturers claim it will destroy 65% of all pests. Between the weekly sprayings it is estimated that 500 new pests invade the trees. A new pesticide, PESTKILL, comes onto the market. The manufacturers claim that it will destroy 85% of existing pests but it is estimated that 650 new pests per week will invade the trees. Which pesticide will be more effective in the long term ? **(7)**

4 Secret Agent 004 has been captured and his captors are giving him a 25 milligram dose of a truth serum every 4 hours. 15% of the truth serum present in his body is lost every hour.

(*a*) Calculate how many milligrams of serum remain in his body after 4 hours (that is immediately **before** the second dose is given). **(3)**

(*b*) It is known that the level of serum in the body has to be continuously above 20 milligrams before the victim starts to confess. Find how many doses are needed before the captors should begin their interrogation. **(3)**

(*c*) Let u_n be the amount of serum (in milligrams) in his body just **after** his n^{th} dose. Show that $u_{n+1} = 0 \cdot 522u_n + 25$. **(1)**

(*d*) It is also known that 55 milligrams of this serum in the body will prove fatal, and the captors wish to keep Agent 004 alive. Is there any maximum length of time for which they can continue to administer this serum and still keep him alive ? **(4)**

5 Biologists calculate that when the concentration of a particular chemical in a sea loch reaches 5 milligrams per litre (mg/l) the level of pollution endangers the life of the fish.
A factory wishes to release waste containing this chemical into the loch. It is claimed that the discharge will not endanger the fish.
The Local Authority is supplied with the following information:

1. The loch contains none of this chemical at present.
2. The factory manager has applied to discharge effluent once per week which will result in an increase in concentration of 2.5 mg/l of the chemical in the loch.
3. The natural tidal action will remove 40% of the chemical from the loch every week.

(*a*) Show that this level of discharge would result in fish being endangered. **(3)**

When this result is announced, the company agrees to install a cleaning process that reduces the concentration of chemical released into the loch by 30%.

(*b*) Show the calculations you would use to check this revised application. Should the Local Authority grant permission ? **(5)**

6 (*a*) At 12 noon a hospital patient is given a pill containing 50 units of antibiotic.
By 1 pm the number of units in the patient's body has dropped by 12%.
By 2 pm a further 12% of the units remaining in the body at 1 pm is lost.
If this fall-off rate is maintained, find the number of units of antibiotic remaining at 6 pm. **(4)**

(*b*) A doctor considers prescribing a course of treatment which involves a patient taking one of these pills every 6 hours over a long period of time.
The doctor knows that mopre than 100 units of this antibiotic in the body is regarded as too dangerous.
Should the doctor prescribe this course of treatment?
Give reasons for your answer. **(6)**

7 The extract below is taken from the "**Oil Rig News**".

RARE ILLNESS STRIKES RIG

Storm prevents delivery of medicine

By noon on Tuesday 20th December 1988 50 of our oil rig personnel were laid low by a mystery illness.
Our resident medical officer is expressing concern because the number of personnel affected is increasing each day by 8% of the previous day's total.

(*a*) If the daily rate of increase remained at 8% of the previous day's total, how many personnel were affected by noon on Sunday 25th December 1988? **(3)**

(*b*) An improvement in the weather conditions allowed a team of medics to fly out to the rig on the morning of Tuesday 27th December 1988.
At noon on that Tuesday, all personnel were innoculated and no new cases of the illness arose. Within the next 24 hours, 21% of those who had been affected had recovered.
If the daily rate of recovery of 21% of the previous day's total was maintained, how many personnel were still affected by the illness at noon on Saturday 31st December 1988? **(5)**

8

Some environmentalists are concerned that the presence of chemical nitrates in drinking water presents a threat to health.

The World Health Organisation recommends an upper limit of 50 milligrams per litre (mg/l) for nitrates in drinking water, although it regards levels up to 100 mg/l as safe.

A sub-committee of a Local Water Authority is considering a proposal affecting a small loch which supplies a nearby town with drinking water. The proposal is that a local factory be permitted to make a once-a-week discharge of effluent into the loch, provided that a cleaning treatment of the loch is carried out before each discharge of effluent.

The Water Engineer has presented the following data:

1. The present nitrate level in the loch is 20 mg/l.
2. The cleaning treatment removes 55% of the nitrates from the loch.
3. Each discharge of effluent will result in an addition of 26 mg/l to the nitrate presence in the loch.

and advises the sub-committee that the proposal presents no long-term danger from nitrates to the drinking water supply.

(*a*) Show the calculations you would use to check the engineer's advice. **(5)**

(*b*) Is the engineer's advice acceptable? **(1)**